ON EXCURSIONS OF REFLECTING BROWNIAN MOTION

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ABSTRACT. We discuss the properties of excursions of reflecting Brownian motion on a bounded smooth domain in R^d and give a procedure for constructing the process from the excursions and the boundary process. Our method is computational and can be applied to general diffusion processes with reflecting type boundary conditions on compact manifolds.

1. **Introduction.** We study the excursions of reflecting Brownian motion (RBM) $X = \{X_t, t \geq 0\}$ on a bounded domain D in R^d with smooth boundary. The excursion structure of one-dimensional RBM has been studied by many authors ([**2**, **6**, **7**, **9**], to mention just a few). It is therefore interesting to see whether the elegant methods developed for this case can be generalized to higher-dimensional cases. The difficulties in doing this arise mainly from the fact that in the latter cases the excursion set is no longer a single point but the whole boundary; as a result we have to consider the so-called boundary process $\xi_t = X_{\tau(t)}$, where $\tau(t)$ is the right continuous inverse of the boundary local time of X. One immediately faces the problem of constructing sample paths of RBM by attaching the individual excursions to the boundary process. §7 deals with this problem. The main idea, due to Itô, is to regard the excursions as a point process. Unlike the one reflecting barrier case, however, the point process of excursions is no longer Poisson.

We will assume throughout that D is a bounded domain with C^3 boundary. This strong regularity assumption enables us to keep the necessary estimates from analysis to a minimum and to study the excursion structure by explicit computation, which is the main feature of the method adopted here. Although some of our results are known in more abstract terms in the general excursion theory of Markov processes ([**4**, **8**, **12**], etc.), our approach here is completely independent of this theory.

Our method is perfectly generalizable to general diffusions on a compact Riemannian manifold with reflecting type boundary condition. We will make more comments on this point in §9. We choose the RBM because, on the one hand, it is a typical case of diffusion revealing the general structure of excursions, and on the other hand, it is well adapted to explicit computation.

We give a short outline of the contents of each section. The basic properties of RBM on a bounded domain are reviewed in §2. In §3–6 we study the excursion processes and the boundary process associated with the RBM. For each pair of distinct points a, b on the boundary, we introduce in §3 a probability law $P^{a,b}$

which governs the excursion process from a to b. The characterizing property of the law $P^{a,b}$ is given in Theorem 3.5.

The boundary process $\{\xi_t, t \geq 0\}$ is a ∂D-valued strong Markov process of jump type. The properties of the boundary process are studied in §4. We limit ourselves to those properties which are needed in subsequent sections. In §5, the point process of excursions of the RBM is defined and studied. We prove that this point process possesses a compensating measure of the form $Q^{\xi_t}(de)\,dt$, where Q^a, called the excursion law from a, is a σ-finite measure on the space of excursions starting from $a \in \partial D$. The form of the compensating measure was known before, but it seems that a rigorous verification was never given. As an application of excursion laws, we prove in §6 a theorem concerning the asymptotic number of excursions and the boundary local time, whose one-dimensional version is well known.

§7 is the center of our discussion. To describe our result in a heuristic way, what we show is that one can construct the RBM paths by filling the jumps of the boundary process with excursion paths. The key to the construction is the observation that, conditioned on the boundary process, the individual excursions are independent. We first construct a point process of excursions and show that it has the desired probabilistic structure. The sample paths of process to be constructed are defined by this excursion process in the usual way (see, for example, [**6**, pp. 123–131]). The last step involves verifying that the process constructed is indeed the RBM.

§8 contains a sketch of proofs of several estimates used in the previous sections and is included here for the sake of completeness. In §9, we make some further remarks on the problem treated in this paper.

2. Reflecting Brownian motion. Let D be a bounded domain in R^d ($d \geq 2$) with C^3 boundary. The Laplace operator and the *inward* normal derivative at the boundary ∂D will be denoted by Δ and $\partial/\partial n$. Roughly speaking, the RBM on D is the diffusion process generated by the operator $\Delta/2$ with the Neumann (reflecting) boundary condition. We will give three equivalent definitions of this diffusion process.

Consider first the heat equation with the Neumann boundary condition:

$$(2.1) \quad \begin{cases} \dfrac{\partial}{\partial t} p(t,x,y) = \dfrac{1}{2}\Delta_x p(t,x,y), & x \in D, \quad y \in \overline{D}; \\[2mm] \dfrac{\partial}{\partial n_x} p(t,x,y) = 0, & x \in \partial D, \ y \in \overline{D}; \\[2mm] \lim_{t \to 0} p(t,x,y) = \delta_y(x), & x \in \overline{D}, \quad y \in \overline{D}. \end{cases}$$

(Here the subscript x means the operation is performed on x variables.) If D has C^3 boundary, this equation can be solved by the parametrix method. On the canonical path space based on $\overline{D}$, we can construct a continuous strong Markov process (a diffusion process) $\{C([0,\infty), \overline{D}), \mathcal{F}, \mathcal{F}_t, P^x, X_t, t \geq 0\}$ whose transition density function is $p(t,x,y)$, and we define this process to be the standard RBM on D. See [**13**] for details.

We can also define RBM by a stochastic differential equation with boundary condition. Consider the following equation with reflecting boundary condition:

$$(2.2) \qquad dX_t = dB_t + \tfrac{1}{2}n(X_t)\, d\phi(t),$$

where $B = \{B_t, t \geq 0\}$ is standard Brownian motion in R^d and $n(a)$ denotes the inward unit normal vector at $a \in \partial D$. The factor $\tfrac{1}{2}$ is added for technical reasons. Now if the domain D is assumed to be C^2, a solution to (2.2) exists and pathwise uniqueness holds [6]. The existence and pathwise uniqueness of the solution to (2.2) can be proved via the so-called deterministic Skorohod equation, without resorting to the general theory of stochastic differential equations [5].

The increasing process ϕ in (2.2) is a continuous additive functional of X and is called the boundary local time of the RBM. It can also be characterized as the unique continuous additive functional satisfying

$$(2.3) \qquad E^x[\phi(t)] = \int_0^t ds \int_{\partial D} p(s,x,b)\, \sigma(db),$$

where σ denotes the $(d-1)$-dimensional volume measure on ∂D. Using Itô's formula we can show that

$$(2.4) \qquad \phi(t) = \lim_{\varepsilon \to 0} \int_0^t \frac{I_{D_\varepsilon}(X_s)}{\varepsilon}\, ds.$$

Here $D_\varepsilon = \{x \in \overline{D} \colon d(x, \partial D) \leq \varepsilon\}$. The convergence is in the sense of L^2 for each fixed t as well as almost surely uniform on bounded intervals of time.

To show that the previous two definitions give the same process, we state yet a third definition—a martingale characterization of RBM. Let $(C([0,\infty), \overline{D}), \mathcal{F}, \mathcal{F}_t)$ be the usual continuous path space based on $\overline{D}$ equipped with the standard filtration. We say that a probability measure P^x on this space is an RBM on D starting at $x \in \overline{D}$ if $P^x[X_0 = x] = 1$ and it is a solution to the submartingale problem of $(\Delta, \partial/\partial n)$; namely, for any $f \in C^2(\overline{D})$ with $\partial f/\partial n \geq 0$, the process

$$(2.5) \qquad M_f(X_t) = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta f(X_s)ds$$

is an $\mathcal{F}_t$-submartingale. By general theory, the RBM in the sense exists, is unique and is a strong Markov process [6, pp. 203–218].

Now all three definitions are equivalent. This follows from the uniqueness of the martingale characterization. The processes in the first and second definitions can be shown easily (in the first case by direct computation, in the second case by Itô's formula) to be solutions of the submartingale problem.

From the second definition above, we have the following extension of the one-dimensional Skorohod theorem [6]. In the one-dimensional case, we simply define RBM by $X_t = |B_t|$ for an ordinary Brownian motion B. This definition has no higher-dimensional analogue.

THEOREM 2.1.   *Let $\{X_t, B_t, \phi(t),\ t \geq 0\}$ be three continuous, $\mathcal{F}_t$-adapted stochastic processes on a certain probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, P\}$ such that the following conditions hold:* (a) $X = (X_t, t \geq 0)$ *is $\overline{D}$-valued and $P[X_0 \in \overline{D}] = 1$;* (b) $B = \{B_t, t \geq 0\}$ *is a standard Brownian motion in R^d with initial value D, and B and*

X_0 are independent; (c) almost surely, ϕ is nondecreasing, $\phi(0) = 0$, and increases only when $X_t \in \partial D$; (d) almost surely, the Skorohod equation holds:

$$(2.6) \qquad X_t = X_0 + B_t + \frac{1}{2} \int_0^t n(X_s)\phi(ds).$$

Then X is a standard RBM on D.

Now assume that $X = \{X_t, t \geq 0\}$ is an RBM. Define the stopping time

$$(2.7) \qquad \tau_D = \inf\{t > 0 : X_t \in \partial D\}.$$

The killed Brownian motion $X^0 = \{X_t, t < \tau_D\}$ is the RBM (also the ordinary Brownian motion) stopped at time τ_D. X^0 is sometimes called the minimal part of X on D. By the strong Markov property at τ_D, we find its transition density function to be

$$(2.8) \qquad p_0(t, x, y) = p(t, x, y) - E^x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t].$$

The distribution of X_{τ_D} is concentrated on ∂D and is absolutely continuous with respect to the volume measure σ on the boundary. The Poisson kernel $K(y, b)$ is defined by

$$(2.9) \qquad P^y[X_{\tau_D} \in db] = K(y, b)\sigma(db).$$

The transition density function (2.8) satisfies the heat equation with absorbing boundary condition:

$$(2.10) \qquad \begin{aligned} &\frac{\partial}{\partial t} p_0(t, x, y) = \frac{1}{2}\Delta_x p_0(t, x, y), & x \in D, \ y \in D, \ t > 0; \\ &p_0(t, x, y) = 0, & x \in \partial D, \ y \in D, \ t > 0; \\ &\lim_{t \to 0} p_0(t, x, y) = \delta_y(x), & x \in D, \ y \in D. \end{aligned}$$

As with equation (2.1), this equation has a unique solution which defines a continuous strong Markov process on D with finite lifetime τ_D. §8 contains a sketch of the construction of $p_0(t, x, y)$ by the parametrix method.

In the next section, we need the joint distribution of (X_{τ_D}, τ_D).

THEOREM 2.2. *We have for $t > 0$, $x \in D$, and $b \in \partial D$,*

$$(2.11) \qquad P^x[X_{\tau_D} \in db, \tau_D \in dt] = g(t, x; b)\sigma(db) \, dt$$

where

$$(2.12) \qquad g(t, x; b) = \frac{1}{2} \frac{\partial}{\partial n_b} p_0(t, x, b).$$

A proof can be found in [1, p. 261].

3. Excursion processes. Let $X = \{X_t, t \geq 0\}$ be the standard RBM on D with probabilities P^x. Fix a time $t > 0$. Define two random variables $\gamma(t)$ and $\beta(t)$ as follows:

$$\gamma(t) = \sup\{s \leq t : X(s) \in \partial D\},$$

$$\beta(t) = \inf\{s \geq t : X(s) \in \partial D\}.$$

As usual, we set $\sup \varnothing = -\infty$. Since $P^x[X_t \in D] = 1$, we have

$$P^x[\gamma(t) < t < \beta(t) < \infty] = 1.$$

Following [2], we call the random interval $[\gamma(t), \beta(t)]$ an excursion interval and the process $Z(u) = X(\gamma(t) + u)$, $0 \le u \le \beta(t) - \gamma(t)$ the excursion process straddling t. The length of the excursion is $l(t) = \beta(t) - \gamma(t)$. In this section we compute the law of this excursion process. Our method is an extension of the one used in [2] for one-dimensional Brownian motion.

Given any two distinct points a, b on the boundary, let $W^{a,b}$ be the space of all continuous excursions from a to b; namely, the space of continuous paths $e: [0, \infty) \to \overline{D}$ with the property that $e(0) = a$ and there is an $l > 0$ such that $e(t) \in D$ if $0 < t < l$ and $e(t) = b$ if $t \ge l$. The space $W^{a,b}$ is equipped with a natural filtration

$$\mathcal{B}_t(W^{a,b}) = \sigma\{e(s), \ e \in W^{a,b}, \ 0 \le s \le t\}.$$

We want to define a law $P^{a,b}$ on $W^{a,b}$, the law of the excursion process from a to b. $P^{a,b}$ can be described by the following characterizing property: The law of the excursion process $Z(u) = X(\gamma(t) + u)$ straddling t conditioned by the event $\{\gamma(t) = s, \ X_{\gamma(t)} = a, \ X_{\beta(t)} = b\}$ is $P^{a,b}$ conditioned by $l > t - s$. It is not difficult to guess what $P^{a,b}$ should be. If $e = \{e(t), \ t \ge 0\}$ is the corresponding process, then once $e(t)$ is inside D, it behaves like an ordinary Brownian motion conditioned to exit at point b. Thus the transition density function of $P^{a,b}$ should be

$$(3.1) \qquad p^{a,b}(s, x, t, y) = p_0(t - s, x, y)\frac{K(y, b)}{K(x, b)}.$$

Next, the absolute distribution $P^{a,b}[e(u) \in dy, \ u < l]$ is formally the limit of (3.1) as $x \to a$ and $s \to 0$; thus we have

$$(3.2) \qquad P^{a,b}[e(u) \in dy, \ u < l] = 2g(u, y; a)\frac{K(y, b)}{\partial K(a, b)/\partial n_a}.$$

The probability $P^{a,b}$ is uniquely determined by these two conditions. We can check easily that $P^{a,b}$ defines a homogeneous diffusion process with sample paths in $W^{a,b}$. Let us now show that the $P^{a,b}$ has the characterizing property described above.

Introduce the notation

$$(3.3) \qquad \theta(t, a, b) = \frac{1}{4}\frac{\partial}{\partial n_a}\frac{\partial}{\partial n_b}p(t, a, b).$$

It can be checked easily that for any $0 < s < t$,

$$(3.4) \qquad \theta(t, a, b) = \int_D g(t - s, x; a)g(s, x; b) \, m(dx).$$

From (2.9) and (2.11) we have

$$(3.5) \qquad K(x, b) = \int_0^\infty g(l, x; b)dl,$$

and

$$(3.6) \qquad N(a, b) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial}{\partial n_a}K(a, b) = \int_0^\infty \theta(l, a, b)dl.$$

Now (3.1) and (3.2) become

$$(3.7) \qquad p^{a,b}(s,x,t,y) = p_0(t-s,x,y) \frac{\int_0^\infty g(l,y;b)dl}{\int_0^\infty g(l,x;b)dl},$$

and

$$(3.8) \qquad P^{a,b}[e(u) \in dy, \ u < l] = g(u,y;a) \frac{\int_0^\infty g(l,y;b)dl}{\int_0^\infty \theta(l,a,b)dl} \ m(dy).$$

Integrating the last density over D and using (3.4), we have

$$(3.9) \qquad P^{a,b}[u < l] = \frac{\int_u^\infty \theta(l,a,b)dl}{\int_0^\infty \theta(l,a,b)dl}.$$

This gives the probabilistic meaning of $\theta(t,a,b)$.

PROPOSITION 3.1. *Let t be fixed. Let $u < t < s$, $u < w_1 < w_2 < \cdots < w_n < s$ and $y_i \in D$, $i = 1, \ldots, n$. We have*

$$P^x[\gamma(t) \in du, \ X_{\gamma(t)} \in da, \ X_{w_i} \in dy_i, \ i = 1, \ldots, n, \ X_{\beta(t)} \in db, \ \beta(t) \in ds]$$
$$= du p(u,x,a)\sigma(da)g(w_1 - u, y_1; a)m(dy_1)$$
$$\times \prod_{i=2}^n p_0(w_i - w_{i-1}, y_{i-1}, y_i)m(dy_i)\sigma(db)g(s - w_n, y_n; b)ds.$$

PROOF. We will prove this formula for $n = 2$ and $w_1 < t < w_2$. By the Markov property, conditioned on $X(t)$, the processes before t and after t are independent [3, p. 2]. Therefore, for any nonnegative bounded continuous functions ψ_1, ψ_2 on $[0,\infty)$, ϕ_1, ϕ_2 on ∂D and ξ_1, ξ_2 on D, we have

$$E^x[\psi_1(\gamma(t))\phi_1(X_{\gamma(t)})\xi_1(X_{w_1})\xi_2(X_{w_2})\phi_2(X_{\beta(t)})\psi_2(\beta(t))]$$
$$(3.10) \qquad = \int_D E^x[\psi_1(\gamma(t))\phi_1(X_{\gamma(t)})\xi_1(X_{w_1})|X_t = y]$$
$$\times E^x[\psi_2(\beta(t))\phi_2(X_{\beta(t)})\xi_2(X_{w_2})|X_t = y]p(t,x,y)m(dy).$$

(Convention: $\psi(-\infty) = 0$.) Let us compute the expectations under the integral sign. Using the Markov property at t and $t - w_2$, and Theorem 2.2, we have

$$E^x[\psi_2(\beta(t))\phi_2(X_{\beta(t)})\xi_2(X_{w_2})|X_t = y]$$
$$= E^y[\psi_2(\tau_D + t)\phi_2(X_{\tau_D})\xi_2(X_{w_2-t}); w_2 - t < \tau_D]$$
$$= E^y[\xi_2(X_{w_2-t})E^{X_{w_2-t}}[\psi_2(\tau_D + w_2)\phi_2(X_{\tau_D})]; w_2 - t < \tau_D]$$
$$(3.11) \qquad = \int_D \xi_2(y_2)p_0(w_2 - t, y, y_2)m(dy_2)$$
$$\times \int_0^\infty \psi_2(s + w_2)ds \int_{\partial D} \phi_2(b)g(s,y_2;b)\sigma(db).$$

Next, under the probability $P^x[\cdot\,|X_t = y]$, the law of the reversed process

$$Y = \{Y_u = X_{t-u}, 0 \le u \le t\}$$

is equal to the law of RBM starting at y and conditioned by $Y_t = x$. Denote this law by $E^{y,x;t}$. We have by the Markov property at $t - w_1$,

(3.12)

$$\begin{aligned}
&E^x[\psi_1(\gamma(t))\phi_1(X_{\gamma(t)})\xi_1(X_{w_1})|X_t = y]\\
&\quad = E^{y,x;t}[\psi_1(t - \tau_D)\phi_1(Y_{\tau_D})\xi_1(Y_{t-w_1}); t - w_1 \le \tau_D < t]\\
&\quad = E^{y,x;t}[\xi_1(Y_{t-w_1})E^{Y_{t-w_1},x;w_1}[\psi_1(w_1 - \tau_D)\phi_1(Y_{\tau_D}); \tau_D < w_1]; t - w_1 < \tau_D]\\
&\quad = \int_D \xi_1(y_1)p_0(t - w_1, y, y_1)\frac{p(w_1, y_1, x)}{p(t, y, x)}m(dy_1)\\
&\qquad \times \int_0^{w_1}\psi_1(w_1 - u)du\int_{\partial D}\phi_1(a)g(u, y_1; a)\frac{p(w_1 - u, a, x)}{p(w_1, y_1, x)}\sigma(da).
\end{aligned}$$

It follows from (3.10) to (3.12) that

$$\begin{aligned}
&E^x[\psi_1(\gamma(t))\phi_1(X_{\gamma(t)})\xi_1(X_{w_1})\xi_2(X_{w_2})\phi_2(X_{\beta(t)})\psi_2(\beta(t))]\\
&\quad = \int_0^{w_1}\psi_1(u)du\int_{\partial D}\phi_1(a)p(u, x, a)\sigma(da)\\
&\qquad \times \int_D \xi_1(y_1)g(w_1 - u, y_1; a)m(dy_1)\\
&\qquad \times \int_D \xi_2(y_2)p_0(w_2 - w_1, y_1, y_2)m(dy_2)\\
&\qquad \times \int_{\partial D}\phi_2(db)\int_{w_2}^{\infty}\psi_2(s)g(s - w_2, y_2; b)ds,
\end{aligned}$$

which is equivalent to our assertion.

PROPOSITION 3.2. *Let t be fixed and $Z(v) = X_{\gamma(t)+v}$ the excursion process straddling t. Let $0 < v_1 < v_2 < \cdots < v_n$, $s > t \vee (u + v_n)$ and $y_i \in D$, for $i = 1, \ldots, n$. We have*

$$\begin{aligned}
&P^x[\gamma(t) \in du, \ X_{\gamma(t)} \in da, \ Z(v_i) \in dy_i, \ i = 1, \ldots, n, \ X_{\beta(t)} \in db, \ \beta(t) \in ds]\\
&\quad = dup(u, x, a)\sigma(da)g(v_1, y_1; a)m(dy_1)\\
&\qquad \times \prod_{i=2}^{n}p_0(v_i - v_{i-1}, y_{i-1}, y_i)m(dy_i)\sigma(db)g(s - u - v_n, y_n; b)ds.
\end{aligned}$$

PROOF. Again we assume $n = 2$. Let ϕ_i, ψ_i, ξ_i, $i = 1, 2$, be the same as before. Let $t_{nk} = k/2^n$ and $I_{nk} = (t_{nk}, t_{nk+1}]$. By the preceding proposition and the

dominated convergence theorem,

$$E^x[\psi_1(\gamma(t))\phi_1(X_{\gamma(t)})\xi_1(Z(v_1))\xi_2(Z(v_2))\phi_2(X_{\beta(t)})\psi_2(\beta(t))]$$

$$= \lim_{n \to \infty} \sum_{k=0}^{\infty} E^x[\psi_1(\gamma(t))\phi_1(X_{\gamma(t)})\xi_1(X(t_{nk}+v_1))$$

$$\times \xi_2(X(t_{nk}+v_2))\phi_2(X_{\beta(t)})\psi_2(\beta(t));$$

$$\gamma(t) \in I_{nk} \cap [0,t], \ \beta(t) > t \vee (u+v_2)]$$

$$= \lim_{n \to \infty} \sum_{k=0}^{\infty} \int_{I_{nk}\cap[0,t]} \psi_1(u)du \int_{\partial D} \phi_1(a)p(u,x,a)\sigma(da)$$

$$\times \int_D \xi_1(y_1)g(t_{nk}+v_1-u,y_1;a)m(dy_1) \int_D \xi_2(y_2)p_0(v_2-v_1,y_1,y_2)m(dy_2)$$

$$\times \int_{\partial D} \phi_2(b)\sigma(db) \int_{t\vee(u+v_2)}^{\infty} \psi_2(s)g(s-t_{nk}-v_2,y_2;b)ds$$

$$= \int_0^t \psi_1(u)du \int_{\partial D} \phi_1(a)p(u,x,a)\sigma(da) \int_D \xi_1(y_1)g(v_1,y_1;a)m(dy_1)$$

$$\times \int_D \xi_2(y_2)p_0(v_2-v_1,y_1,y_2)m(dy_2) \int_{\partial D} \phi_2(b)\sigma(db)$$

$$\times \int_{t\vee(u+v_2)}^{\infty} \psi_2(s)g(s-u-v_2,y_2;b)ds,$$

which is exactly what we want.

Take $n = 1$ in the preceding proposition and integrate out y_1 and s. By (3.4), we have

COROLLARY 3.3. *Let* $0 < u < t$. *Then*

$$P^x[\gamma(t) \in du, \ X_{\gamma(t)} \in da, \ X_{\beta(t)} \in db] = dup(u,x,a)\sigma(da) \int_{t-u}^{\infty} \theta(s,a,b)ds \ \sigma(db).$$

As an immediate consequence, we have

THEOREM 3.4. *The excursion process straddling* t

$$Z = \{Z(u) = X(\gamma(t)+u), \ 0 \le u \le l(t)\}$$

conditioned on the event $\{\gamma(t) = s, \ X_{\gamma(t)} = a, \ X_{\beta(t)} = b\}$ *is a nonhomogeneous Markov process with transition density function*

(3.13)
$$p_0(v-u,x,y)\frac{\int_{v\vee(t-s)-v}^{\infty} g(l,y;b)dl}{\int_{u\vee(t-s)-u}^{\infty} g(l,y;b)dl}$$

and absolute distribution

$$P^x[Z(u) \in m(dy), \ u < l(t)|\gamma(t) = s, \ X_{\gamma(t)} = a, \ X_{\beta(t)} = b]$$

(3.14)
$$= g(u,y;a)\frac{\int_{u\vee(t-s)-u}^{\infty} g(l,y;b)dl}{\int_{t-s}^{\infty} \theta(l,a,b)dl}.$$

The law in Theorem 3.4 depends on t because we have only included the excursions from a to b straddling a fixed time t. Thus we expect the "true" law $P^{a,b}$ should be the limit of the law in the theorem as $t \to s$. This agrees with our previous definition of $P^{a,b}$.

THEOREM 3.5. *Let $P^{a,b}$ be defined by (3.1) and (3.2). The law of the excursion process straddling t conditioned on the event $\{\gamma(t) = s,\ X_{\gamma(t)} = a,\ X_{\beta(t)} = b\}$ is equal to $P^{a,b}[\cdot\,|\,l > t - s]$.*

PROOF. By (3.8) to (3.9),

$$\frac{1}{m(dy)} P^{a,b}[e(u) \in dy,\ u < l\,|\,l > t - s]$$

$$= P^{a,b}[e(u) \in dy,\ u \vee (t - s) < l]/P^{a,b}[l > t - s]$$

$$= g(u, y; a) \frac{\int_0^\infty g(l, y; b)dl}{\int_0^\infty \theta(l, a, b)dl} \frac{\int_{u \vee (t-s)-u}^\infty g(l, y; b)dl}{\int_0^\infty g(l, y; b)dl} \Big/ \frac{\int_{t-s}^\infty \theta(l, a, b)dl}{\int_0^\infty \theta(l, a, b)dl}$$

$$= g(u, y; a) \frac{\int_{u \vee (t-s)-u}^\infty g(l, y; b)dl}{\int_{t-s}^\infty \theta(l, a, b)dl}.$$

This agrees with (3.14). A similar calculation shows that (3.13) is the transition density function of $P^{a,b}$ conditioned on $l > t - s$.

Finally we remark that $P^{a,b}$ is completely determined by the minimal part of the RBM X and has nothing to do with the boundary condition. Using RBM to characterize this law is simply a matter of convenience.

4. The boundary process.

Loosely speaking, the boundary process is the trace of the RBM X on the boundary. Since X spends zero amount of time on the boundary, a time change is needed here. Recall that the boundary local time ϕ (see (2.4)) is a continuous strong additive functional which increases only when $X_t \in \partial D$. Let

$$(4.1) \qquad\qquad \tau(t) = \sup\{s \geq 0 \colon \phi(s) \leq t\}$$

be the right continuous inverse of ϕ. We have $\tau(\phi(t)-) \leq t \leq \tau(\phi(t))$ and $\phi(\tau(t)) = t$. For fixed t, each $\tau(t)$ is a stopping time for X. The boundary process is defined by

$$(4.2) \qquad\qquad \xi = \{\xi_t = X_{\tau(t)}, t \geq 0\}.$$

It is routine to verify that ξ is a ∂D-valued strong Markov process and the sample paths of ξ are right continuous with left limits. In this section, we discuss a few properties of the boundary process needed later.

Let us first describe the infinitesimal generator of ξ. Let $f \in C^{2,\alpha}(\partial D)$ and let u_f be the solution of the Dirichlet problem on D with boundary function f. Then by the boundary regularity theorem, we have $u_f \in C^{2,\alpha}(\overline{D})$. Define the operator

$$Af(a) = \frac{\partial u_f}{\partial n}(a), \qquad a \in \partial D.$$

Thus A is well defined on $C^{2,\alpha}(\partial D)$ and is called the Dirichlet-Neumann operator of D (for the obvious reason). A is in general an integrodifferential operator on ∂D.

PROPOSITION 4.1. *The restriction to $C^{2,\alpha}(\partial D)$ of the infinitesimal generator in $C(\partial D)$ of the boundary process ξ is $\frac{1}{2}A$.*

PROOF. Assume that $\{\mathcal{F}_t, t \geq 0\}$ is the natural filtration associated with the RBM X. Let $f \in C^{2,\alpha}(\partial D)$, and u_f as before. By the Skorohod equation (2.6), we

have

$$u_f(X_t) - u_f(X_0) = \int_0^t \nabla u_f(X_s) \cdot dB_s + \frac{1}{2} \int_0^t \nabla u_f \cdot n(X_s)d\phi(s).$$

We assume that $X_0 \in \partial D$. Since $\tau(t)$ is a stopping time, we can replace t in the above relation by $\tau(t)$ and obtain

$$f(\xi_t) - f(\xi_0) = \int_0^{\tau(t)} \nabla u_f(X_s) \cdot dB_s + \frac{1}{2} \int_0^t Af(\xi_s)ds.$$

The first term on the right side is obviously uniformly bounded for fixed t; hence it is a $\mathcal{F}_{\tau(t)}$-martingale. The assertion follows by taking the expectation.

Let us now obtain a more explicit description of the Dirichlet-Neumann operator A. For each $b \in \partial D$, let V_b be a C^2 vector field on ∂D with the following properties: (i) for any fixed $a \in \partial D$ and $f \in C^2(\partial D)$, we have

(4.3) $$f(b) - f(a) - V_b f(a) = O(||b - a||^2)$$

as $b \to a$. (ii) If $\nabla f(a) = 0$, then $V_b f(a) = 0$ for all b. It is not difficult to verify the existence of V_b. For example, in a coordinate neighborhood $b = (b_1, \ldots, b_{d-1})$, one may take

$$V_b f(a) = \sum_{i=1}^{d-1} \frac{\partial f}{\partial b_i}(a)(b_i - a_i),$$

and then obtain V_b globally by a partition of unity.

Now outside a neighborhood of a, the Poisson kernel $K(x, a)$ is well-behaved. Thus if $f \in C^2(\partial D)$ vanishes in a neighborhood of a, then we can write

(4.4) $$Af(a) = \frac{\partial}{\partial n_a} \int_{\partial D} K(a, b)f(b)\sigma(db) = 2 \int_{\partial D} f(b)N(a, b)\sigma(db).$$

On the other hand, by (8.2), we have $N(a, b) = O(||a - b||^{-d})$; hence (4.3), the operator

(4.5) $$A_0 f(a) = 2 \int_{\partial D} [f(b) - f(a) - V_b f(a)]N(a, b)\sigma(db)$$

is well defined on $C^2(\partial D)$. The operator $D_1 = A - A_0$ is a local operator in the sense that if f vanishes in a neighborhood of a, then $D_1 f(a) = 0$. This is true because for such f, the right side of (4.5) reduces to that of (4.4). By a theorem in analysis, a local operator is a differential operator. Hence D_1 is a differential operator. D_1 is even a vector field, since $D_1 f(a) = 0$ if $f(a) = 0$ and $\nabla f(a) = 0$. Therefore we have shown

PROPOSITION 4.2. *The Dirichlet-Neumann operator has the form*

(4.6) $$A = A_0 + D_1,$$

where D_1 is a vector field on ∂D and A_0 is given as in (4.5).

It follows that A is an integrodifferential operator without diffusion part, and ξ is the process of jump type generated by the operator $\frac{1}{2}A$. The existence of processes generated by integrodifferential operators is discussed in [10].

Let us look more closely at the jumps of ξ. Let J be the set of times when ξ has a jump:

$$(4.7) \qquad\qquad J = \{s \geq 0 : \xi_{s-} \neq \xi_s\}.$$

We also set

$$J_t = J \cap (0, t] \quad \text{and} \quad J_{t-} = J \cap (0, t)$$

to simplify notation.

An immediate consequence of Proposition 4.2 is

PROPOSITION 4.3. *Let ξ be the process on ∂D generated by the operator $\frac{1}{2}A$. Then the Lévy system of ξ is $(N(a,b)\sigma(db), dt)$. Namely, for any nonnegative measurable function f on $\partial D \times \partial D$ and any stopping time τ of ξ, we have*

$$(4.8) \qquad E^a\left[\sum_{s \in J_\tau} f(\xi_{s-}, \xi_s)\right] = E^a\left[\int_0^\tau ds \int_{\partial D} f(\xi_s, b) N(\xi_s, b)\sigma(db)\right].$$

See [10] for a proof of (4.8).

For later discussion, we also need

PROPOSITION 4.4. (a) *Almost surely, the random set J is dense on $R^+ = (0, \infty)$.* (b) *There exists a constant $\varepsilon > 0$ depending only on ∂D, such that almost surely,*

$$\limsup_{\substack{t \to \infty \\ t \in J}} \|\xi_t - \xi_{t-}\| \geq \varepsilon.$$

PROOF. (a) Let

$$\tau_t = \inf\{s > t : \xi_{s-} \neq \xi_s\}.$$

Clearly, τ_t is a stopping time of ξ. Letting $f \equiv 1$ and $\tau = \tau_0$ in (4.8) and using $\int_{\partial D} N(a,b)\sigma(db) = \infty$, we obtain $1 = \infty \cdot E^a[\tau_0]$. Hence $\tau_0 = 0$ a.s. It follows from the Markov property that we have $\tau_t = 0$ a.s. simultaneously for all rational t, which implies that J is dense in R^+.

(b) The proof is similar. Let us choose $\varepsilon > 0$ such that for any $a \in \partial D$, we have $\sigma(B_\varepsilon(a)^c) > 0$, where $B_\varepsilon(a) = \{b \in \partial D : \|a - b\| \leq \varepsilon\}$. Such ε obviously exists. Now let

$$\sigma_t = \inf\{s > t : \|\xi_s - \xi_{s-}\| \geq \varepsilon\}.$$

($\inf \varnothing = \infty$) Let $\tau = \sigma_0$ and $f(a,b) = I_{[\varepsilon, \infty)}(\|a - b\|)$ in (4.8), we have

$$1 \geq E^a\left[\int_0^{\sigma_0} N(\xi_s, B_\varepsilon(\xi_s)^c) ds\right].$$

Since $N(a,b)$ is bounded uniformly away from zero, by the choice of ε, there exists $\delta > 0$ such that $N(a, B_\varepsilon(a)^c) \geq \delta$ for any $a \in \partial D$. It follows from the above equality that $1 \geq \delta E^a[\sigma_0]$; hence $\sigma_0 < \infty$ a.s. Again by the Markov property, we have $\sigma_n < \infty$, $n = 0, 1, \ldots$, which implies part (b).

5. Point process of excursions. RBM gives rise to a point process taking values in the space of excursions. The idea of regarding excursions of a Markov process from a set in the state space as a point process is due to Itô. Let us begin with a brief review of some basic notions of point processes. We follow the exposition of [6]. Suppose $(E, \mathcal{E})$ is a measurable space and $E_\partial = E \cup \{\partial\}$, where

∂ is a fictitious point attached to E. An E-valued point function is a measurable function $e\colon [0,\infty) \to E_\partial$ such that the set $J(e) = \{s > 0\colon e_s \in E\}$ is countable. For each point function e, the counting measure n_e is defined by

$$n_e(C) = \{(s,x) \in C\colon e_s = x\}$$

for $C \subset (0,\infty) \times E$. We denote the set of E-valued point functions by $\Gamma(E)$.

Now let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space equipped with a filtration of σ-algebras. A function $e\colon \Omega \mapsto \Gamma(E)$ is called an $\mathcal{F}_t$-adapted point process if for each set $C \in \mathcal{E}$, the increasing process $t \mapsto n_e((0,t] \times C)$ is $\mathcal{F}_t$-adapted.

One of the important characterizations of a point process is its so-called compensating measure.

DEFINITION. *A σ-finite random measure $\hat{n}_e$ on the measurable space $((0,\infty) \times E$, $\mathcal{B}(0,\infty) \times \mathcal{E})$ is called a compensating measure of e if there is a sequence of sets $\{U_n\} \subset \mathcal{E}$ exhausting E with the following properties:*
  (a) *For each n, the function $t \mapsto \hat{n}_e((0,t) \times U_n)$ is continuous;*
  (b) *for each n, we have $E[\hat{n}_e((0,t) \times U_n)] < \infty$;*
  (c) *for any set $V \in \mathcal{E}$ contained in some U_n, the process*

$$t \mapsto \tilde{n}_e((0,t) \times V) = n_e((0,t) \times V) - \hat{n}_e((0,t) \times V)$$

is an $\mathcal{F}_t$-martingale.

A point process possessing a compensating measure is said to be of class (QL) (quasi-left continuous). The compensating measure, if it exists, is unique. As an example, it is a well-known result that if $\hat{n}_e(dx\,dt) = m(dx)dt$ for some σ-finite measure m on E, then e is a Poisson point process whose law can be written down explicitly in terms of m [6, pp. 43–44].

For each point process of class (QL), there is a corresponding stochastic integration theory. We say that a real-valued measurable function $f(t,x,\omega)$ on $[0,\infty) \times E \times \Omega$ is $\mathcal{F}_t$-predictable if for each fixed t, $f(t,x,\omega)$ is $\mathcal{E} \times \mathcal{F}_t$ measurable, and for each fixed x, $(t,\omega) \mapsto f(t,x,\omega)$ is an $\mathcal{F}_t$-predictable process.

Let e be a point process of class (QL). Define

$$I_e = \left\{ \mathcal{F}_t\text{-predictable } f\colon \forall t > 0,\ \int_0^{t+} |f(s,x,\omega)| n_e(dx\,ds) < \infty \text{ a.s.} \right\};$$

$$I_e^1 = \left\{ \mathcal{F}_t\text{-predictable } f\colon \forall t > 0,\ E\left[\int_0^t |f(s,x,\omega)| \hat{n}_e(dx\,ds)\right] < \infty \right\};$$

$$I_e^2 = \left\{ \mathcal{F}_t\text{-predictable } f\colon \forall t > 0,\ E\left[\int_0^t |f(s,x,\omega)|^2 \hat{n}_e(dx\,ds)\right] < \infty \right\}.$$

Then for each $f \in I_e^1 \cap I_e^2 \subset I_e$, we can define the stochastic integrals

(5.1) $$n(f)_t = \int_0^{t+} \int_E f(s,x,\omega) n_e(dx\,ds) = \sum_{s \in J_t} f(s,e_s,\omega),$$

(5.2) $$\hat{n}_e(f)_t = \int_0^{t+} \int_E f(s,x,\omega) \hat{n}_e(dx\,ds),$$

and

(5.3) $$\tilde{n}_e(f)_t = \int_0^{t+} \int_E f(s,x,\omega) \tilde{n}_e(dx\,ds) = n_e(f)_t - \hat{n}_e(f)_t.$$

Here $J_t = J \cap (0, t]$. The process $t \mapsto \tilde{n}_e(f)_t$ is a square integrable $\mathcal{F}_t$-martingale whose quadratic variation process is $t \mapsto \tilde{n}_e(f^2)_t$.

Let us now return to the RBM $\{X_t, \mathcal{F}_t, P^x, t \geq 0\}$ and define the point process associated with it.

Recall that $W^{a,b}$ is the space of continuous excursions from a to b. Let us set

$$W^a = \bigcup_{b \in \partial D, \ b \neq a} W^{a,b}$$

(the space of excursions from a) and

$$W = \bigcup_{a \in \partial D} W^a$$

(the space of all excursions). As with $W^{a,b}$, the spaces W^a and W are equipped with natural filtrations $\mathcal{B}_t(W^a)$ and $\mathcal{B}_t(W)$. As before, let $\phi(t)$ be the boundary local time of X and $\tau(t)$ be its right continuous inverse. The point process of excursions of X is a W-valued point process defined as follows. We set

(5.5) $$J = \{s \in (0, \infty) : \tau(s-) < \tau(s)\}.$$

For each $s \in J$, define $l(s) = \tau(s) - \tau(s-)$ and

(5.6) $$e_s(t) = \begin{cases} X(t + \tau(s-)), & \text{if } t \leq l(s); \\ X(\tau(s)), & \text{if } t \geq l(s). \end{cases}$$

If $s \notin J$, define $e_s = \partial$. It is clear that the point process e thus defined is $\mathcal{G}_t$-adapted with $\mathcal{G}_t = \mathcal{F}_{\tau(t)}$. We call this e the point process of excursions of X.

THEOREM 5.1. *The point process of excursions of RBM X is of class (QL), and its compensating measure is given by*

(5.7) $$\hat{n}((0, t) \times C) = \int_0^t Q^{\xi_s}(C \cap \{e : e(0) = \xi_s\}) ds.$$

where ξ is the boundary process, and the σ-finite measure Q^a on W^a, called the excursion law from a, has transition density function $p_0(t, x, y)$ and absolute distribution

(5.8) $$Q^a[e(t) \in dy; t < l] = g(t, y; a) m(dy).$$

PROOF. There is no question about the existence of Q^a. In fact, Q^a can be defined by

(5.9) $$Q^a[A] = \int_{\partial D} P^{a,b}[A \cap \{e : e(l) = b\}] N(a, b) \sigma(db)$$

for a measurable $A \subset W^a$.

For the sequence of sets $\{U_n\}$ in the definition of compensating measure, we take

$$U_n = \{e \in W : l(e) > 1/n\}.$$

Since the function $l(e)$ is positive on W by definition, we have $U_n \uparrow W$. The increasing process $t \mapsto \hat{n}((0, t) \times U_n)$ is obviously continuous and $\mathcal{G}_t$-adapted. It is integrable because by (5.10) below we have

$$Q^a[U_n] = \int_D g(n^{-1}, y; a) m(dy),$$

and the right side of this equality is bounded uniformly in $a \in \partial D$.

We claim that for any fixed t and any nonnegative measurable function f on W,

$$(5.10) \qquad E^x \left[\sum_{s \in J_t} f(e_s) \right] = E^x \left[\int_0^t Q^{\xi_s}(f) ds \right],$$

where for any measurable function f on W,

$$Q^a(f) = \int_{W^a} f(e) Q^a(de).$$

Let us prove (5.10). Set $I = \{t > 0 : X_t \notin \partial D\}$. Since I is open, we have $I = \bigcup_\alpha I_\alpha$, where $I_\alpha = (l_\alpha, r_\alpha)$ are the maximal open intervals contained in I. For each α, the path e_α defined by

$$e_\alpha(t) = \begin{cases} X(t + l_\alpha), & \text{if } 0 \le t < r_\alpha - l_\alpha, \\ X(r_\alpha), & \text{if } t \ge r_\alpha - l_\alpha \end{cases}$$

is an element in W. Let

$$A_t(f) = \sum_{l_\alpha \le t} f(e_\alpha).$$

We compute the expectation of $A_t(f)$. Denote the excursion process of X straddling t by Z_t. Recall Theorem 3.4, which says that under the probability P^x, the law of Z_t conditioned on the event $\{\gamma(t) = s, X_{\gamma(t)} = a, X_{\beta(t)} = b\}$ is just $P^{a,b}$ conditioned by $l > t - s$. Set $t_{n,k} = k/2^n$ and $I_{n,k} = (t_{n,k}, t_{n,k+1}]$. By the monotone convergence theorem, we have

$$E^x[A_t(f)] = \lim_{n \to \infty} \sum_{k \le 2^n t} E^x[f(Z_{t_{n,k+2}}); \gamma(t_{n,k+2}) \in I_{n,k}]$$

$$= \lim_{n \to \infty} \int_{I_{n,k}} \int_{\partial D} \int_{\partial D} E^{a,b}[f(e)|l > t_{n,k+2} - s]$$

$$\times P^x[\gamma(t_{n,k+2}) \in ds, \ X_{\gamma(t_{n,k+2})} \in da, \ X_{\beta(t_{n,k+2})} \in db]$$

$$(5.11) \qquad = \lim_{n \to \infty} \int_{I_{nk}} ds \int_{\partial D} p(u, x, a)\sigma(da) \int_{\partial D} E^{a,b}[f(e)|l > t_{n,k+2} - s]$$

$$\times \int_{t_{n,k+2} - s}^\infty \theta(l, a, b) dl \sigma(db)$$

$$= \int_0^t ds \int_{\partial D} p(u, x, a)\sigma(da) \int_{\partial D} E^{a,b}[f(e)] N(a, b)\sigma(db)$$

$$= \int_0^t ds \int_{\partial D} p(u, x, a) Q^a(f)\sigma(da).$$

Here we have used Theorem 3.3, Corollary 3.4 and (5.9) exactly in this order. It follows from this and the identity

$$\int_0^t ds \int_{\partial D} p(s, u, a)g(a)\sigma(da) = E^x \left[\int_0^t g(X_s)\phi(ds) \right],$$

which can be proved by (2.4) via an approximation argument, that

$$(5.12) \qquad E^x[A_t(f)] = E^x \left[\int_0^t Q^{X_s}(f)\phi(ds) \right].$$

(5.12) implies in particular that the process $\int_0^t Q^{X_s}(f)\phi(ds)$ is the dual predictable projection of the increasing process $A_t(f)$ (with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$). It follows that for any nonnegative predictable process Z_t, we have

$$(5.13) \qquad E^x\left[\sum_\alpha Z_{l_\alpha}f(e_\alpha)\right] = E^x\left[\int_0^\infty Z_s Q^{X_s}(f)\phi(ds)\right].$$

In particular, we can let $Z = I_{(0,\tau(t)]}$, because $\tau(t)$ is a stopping time. Making a change of variable on the right side of (5.13), we obtain (5.10).

To finish the proof of the theorem, let V be a measurable subset of W contained in some U_n. Then $t \mapsto \hat{n}((0,t) \times V)$ is $\mathcal{G}_t$-adapted and integrable; and so is $t \mapsto \tilde{n}((0,t) \times V) = n((0,t) \times V) - \hat{n}((0,t) \times V)$, and we have $E^x[\tilde{n}((0,t) \times V)] = 0$ by (5.10). From the strong Markov property and the easily verified identity

$$\tilde{n}((s,t) \times V) = \tilde{n}((0,t-s) \times V) \circ \theta_{\tau(s)}$$

for $s < t$ (θ is the shifting operator of X), we obtain

$$\begin{aligned}
E^x[\tilde{n}((0,t) \times V)|\mathcal{G}_s] &= \tilde{n}((0,s) \times V) + E^x[\tilde{n}((0,t-s) \times V) \circ \theta_{\tau(s)}|\mathcal{F}_{\tau(s)}] \\
&= \tilde{n}((0,s) \times V) + E^{X_{\tau(s)}}[\tilde{n}((0,t-s) \times V)] \\
&= \tilde{n}((0,s) \times V).
\end{aligned}$$

Theorem 5.1 is proved.

6. An application of the excursion law. Recall that $l(e)$ is the length of the excursion e. Let A be a measurable subset of ∂D and $l_\varepsilon^A(t)$ be the number of excursions of length not less than ε and that start from A and occur before time t.

As an application of the excursion laws Q^a of the last section, we prove the following theorem whose one-dimensional version is well-known.

THEOREM 6.1. *With probability one, we have for all $t > 0$,*

$$(6.1) \qquad \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, l_\varepsilon^A(t) = \sqrt{\frac{2}{\pi}} \int_0^t I_A(X_s)\phi(ds).$$

PROOF. The key to the proof is the asymptotic formula (8.1). Let $f_\varepsilon^A(e) = I_{\{l \geq \varepsilon\}}(e)I_A(e(0))$. By definition, we have

$$l_\varepsilon^A[\tau(t)] = \int_0^t \int_W f_\varepsilon^A(e)n(de\, ds).$$

Since $f_\varepsilon^A \in I_\varepsilon^1 \cap I_\varepsilon^2$, the process

$$(6.2) \qquad \Delta_\varepsilon^A(t) = l_\varepsilon^A(\tau(t)) - \int_0^t \int_W f_\varepsilon^A(e)Q^{\xi_s}(de)ds = \int_0^t \int_W f_\varepsilon^A(e)\tilde{n}(de\, ds)$$

is a square integrable martingale with quadratic variation process

$$\int_0^t ds \int_W [f_\varepsilon^A]^2(e)Q^{\xi_s}(de) = \int_0^t Q^{\xi_s}[e: l(e) > \varepsilon] \cdot I_A(\xi_s)ds.$$

Therefore by (8.1), we have

$$E|\Delta_\varepsilon^A(t)|^2 = O(t\varepsilon^{-1/2}).$$

It follows from Doob's inequality that for fixed T,

$$\sum_{n=2}^{\infty} P\left[\sqrt{\varepsilon_n} \sup_{0 \le t \le T} |\Delta_{\varepsilon_n}^A(t)| \ge \frac{1}{\log n}\right] \le K \sum_{n=2}^{\infty} \frac{\log^2 n}{n^2} < \infty,$$

where $\varepsilon_n = n^{-4}$. By the Borel-Cantelli lemma, we have

(6.3) $$P\left[\forall t \ge 0: \lim_{n \to \infty} \sqrt{\varepsilon_n}\Delta_{\varepsilon_n}^A(t) = 0\right] = 1.$$

Again by (8.1),

(6.4) $$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \int_0^t \int_W f_\varepsilon^A(e) Q^{\xi_s}(de)ds = \lim_{\varepsilon \to 0} \int_0^t \sqrt{\varepsilon} Q^{\xi_s}[l > \varepsilon] I_A(\xi_s) ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^t I_A(\xi_s)ds.$$

Hence, from (6.2) to (6.4),

(6.5) $$P\left[\forall t \ge 0: \lim_{n \to \infty} \sqrt{\varepsilon_n} l_{\varepsilon_n}^A(\tau(t)) = \sqrt{\frac{2}{\pi}} \int_0^{\tau(t)} I_A(X_s)\phi(ds)\right] = 1.$$

Now for any ε choose $n = [1/\varepsilon]$. We have

$$[1 + 1/n]^{-2} l_{\varepsilon_n}^A(\tau(t)) \le \sqrt{\varepsilon} l_\varepsilon^A(\tau(t)) \le [1 + 1/n]^2 l_{\varepsilon_{n+1}}(\tau(t)).$$

This and (6.5) imply

$$P\left[\forall t \ge 0: \lim_{\varepsilon \to 0} \sqrt{\varepsilon} l_\varepsilon^A(\tau(t)) = \sqrt{\frac{2}{\pi}} \int_0^{\tau(t)} I_A(X_s)\phi(ds)\right] = 1,$$

which is equivalent to (6.1).

7. Construction of the RBM. Since the RBM spends zero amount of time on the boundary, one may think of it intuitively as obtained by piecing together its excursions. In the one-dimensional case, this procedure was carried out in various forms (see [**6, 7, 9**]). One of the approaches is to start with the point process of excursions.

Our intention in this section is to carry out the same procedure for higher-dimensional RBM. This case is more interesting because of the presence of the boundary process. Observing that, conditioned by the boundary process, the individual excursion processes are independent and depend only on the endpoints [**8**], we naturally start with a collection of independent copies of excursion processes $\{W^{a,b}, e_t^{a,b}, \mathcal{F}_t^{a,b}, P^{a,b}\}$, and a jump process ξ on the boundary generated by the Dirichlet-Neumann operator $\frac{1}{2}A$. We have shown that the Lévy system of ξ is $(N(a,b)\sigma(db), dt)$. Since the Lévy system describes the jumps of the process and these jumps come from the excursions, there must be a consistency condition which involves both the Lévy kernel and the excursion laws. This is exactly the meaning of identity (5.9).

The first step in our construction is to obtain a point process of excursions with the expected compensating measure (5.7). Let us first define a probability space on which our processes are based. Let $(\Omega_1, \mathcal{H}, \mathcal{H}_t, \xi_t, P_1^a)$ be the jump process on ∂D generated by the operator $\frac{1}{2}A$, as described in §4. We may assume $\mathcal{H}_t = \sigma\{\xi_s, s \le t\}$

and $\mathcal{H} = \bigvee_{t \geq 0} \mathcal{H}_t$, appropriately completed if necessary. Let $(\Omega_2, \mathcal{M}, P_2)$ be another probability space on which are defined, for each triple $(a, b, i) \in \partial D \times \partial D \times N^+$, $a \neq b$, an excursion process $e^{a,b;i}$ with the law $P^{a,b}$, and we require further that these excursion processes are mutually independent. Finally let

$$(7.1) \qquad (\Omega, \mathcal{G}, P^a) = (\Omega_1 \times \Omega_2, \mathcal{H} \times \mathcal{M}, \ P_1^a \times P_2).$$

A typical element in Ω has the form $\omega = (\omega_1, \omega_2)$.

After these preparations, we now define the point process of excursions. Suppose $\omega \in \Omega$. Let

$$(7.2) \qquad J(\omega) = \{s > 0 \colon \xi_{s-} \neq \xi_s\}.$$

We know that almost surely J is a dense countable subset of R^+. We will call a point $s \in J(\omega)$ a jump time of size (ξ_{s-}, ξ_s). Define a map $e \colon R^+ \times \Omega \to W \cup \partial$ as follows:

$$(7.3) \quad e_s(\omega) = \begin{cases} e^{\xi_{s-}, \xi_s; i}(\omega), & \text{if } s \in J(\omega) \text{ is the } i\text{th jump time of size } (\xi_{s-}, \xi_s); \\ \partial, & \text{if } s \notin J(\omega). \end{cases}$$

Let $\mathcal{G}_t$ be the σ-field generated by the random variables ξ_s, e_s, $s \leq t$. Then the point process e defined above is $\mathcal{G}_t$-adapted.

PROPOSITION 7.1. *The $\mathcal{G}_t$-adapted point process e defined above is of class* (QL) *with the compensating measure $Q^{\xi_s}(de)ds$.*

PROOF. For $\{U_n\}$ in the definition of compensating measure we can still take $U_n = \{e \in W \colon l(e) > 1/n\}$. Let f be a nonnegative measurable function on W so that for any $t > 0$,

$$E^a \left[\int_0^t Q^{\xi_s}(|f|)ds \right] < \infty.$$

We have to show that

$$(7.4) \qquad \tilde{n}(f)_t = \int_0^t \int_W f(e) n(de \, ds) - \int_0^t Q^{\xi_s}(f)ds$$

is a $\mathcal{G}_t$-martingale. Let us check that $\tilde{n}(f)_t$ is integrable. We have by (7.3) and (4.8) that

$$E^a \left[\sum_{u \in J_t} f(e_u) \right] = E_1^a \left[\sum_{u \in J_t} E_2(f(e_u)) \right] = E_1^a \left[\sum_{u \in J_t} E^{\xi_{s-}, \xi_s}(f(e_u)) \right]$$

$$= E_1^a \left[\int_0^t ds \int_{\partial D} E^{\xi_s, b}(f(e)) N(\xi_s, b) \sigma(db) \right].$$

Hence by the consistency condition (5.9),

$$(7.5) \qquad E^a \left[\sum_{u \in J_t} f(e_u) \right] = E^a \left[\int_0^t Q^{\xi_s}(f)ds \right].$$

We now show that for any bounded $g(\omega) \in \mathcal{G}_s$ and $t \geq s$,

$$(7.6) \qquad E[(\tilde{n}(f)_t - \tilde{n}(f)_s)g(\omega)] = 0.$$

By definition, for *fixed* ω_1, under probability P_2, the collection of excursion processes $\{e_u, u \in J_s\}$ is independent of $\{e_u, u \in J_t - J_s\}$. Hence for fixed ω_1 the random variable $g(\omega)$ is independent of

$$\tilde{n}(f)_t - \tilde{n}(f)_s = \sum_{u \in J_t - J_s} f(e_u) - \int_s^t Q^{\xi_s}(f)ds.$$

Consequently, we have

$$E^a[(\tilde{n}(f)_t - \tilde{n}(f)_s)g(\omega)]$$

$$= E_1^a \left\{ E_2[g(\omega)]E_2 \left[\sum_{u \in J_t - J_s} f(e_u) - \int_s^t Q^{\xi_u}(f)ds \right] \right\}$$

$$= E_1^a \left\{ E_2[g(\omega)] \left[\sum_{u \in J_t - J_s} E^{\xi_{u-}, \xi_u}[f(e)] - \int_s^t Q^{\xi_s}(f)ds \right] \right\}.$$

But $E_2[g(\omega)]$ as a function of ω_1, is measurable with respect to $\mathcal{H}_s$. Hence by the Markov property of ξ and (7.5), we obtain

$$E^a[(\tilde{n}(f)_t - \tilde{n}(f)_s)g(\omega)]$$

$$= E_1^a \left\{ E_2[g(\omega)]E_1^{\xi_s} \left[\sum_{u \in J_{t-s}} E^{\xi_{u-}, \xi_u}[f(e)] - \int_0^{t-s} Q^{\xi_u}(f)du \right] \right\}$$

$$= 0.$$

The next step is to get the boundary local time. Recall that l is the lifetime function on Ω. Let

$$(7.7) \qquad\qquad\qquad \tau(t) = \sum_{s \in J_t} l(e_s).$$

By definition, τ is right continuous increasing and $\tau(0) = 0$. To ensure that $\tau(t)$ has a continuous inverse, we need

LEMMA 7.2. *With probability one, the function τ defined above is finite, strictly increasing and tends to infinity as $t \to \infty$.*

PROOF. By Proposition 4.4(a), the set J is dense on $(0, \infty)$. Hence τ is strictly increasing because l is strictly positive on W. Next, we show that $\tau(\infty) = \lim_{t \to \infty} \tau(t) = \infty$ a.s. The function $f(a, b) = E^{a,b}[l(e)]$ is positive and continuous on $\partial D \times \partial D$ with the diagonal deleted. Thus for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(a, b) > \delta$ if $d(a, b) \geq \varepsilon$. It follows by the construction of the point process and Proposition 4.4(b) that for a fixed ω_1, under the probability P_2, the random variable $\tau(\infty)$ is no less than a sum of independent positive random variables with expectations greater than or equal to δ. Hence $\tau(\infty) = \infty$, P_2-a.s. and therefore also P^a-a.s.

The assertion $\tau(t) < \infty$, a.s. for each fixed t is implied by $E[\tau(t)] < \infty$, which we are about to show. From (7.17) below, we have

$$Q^a[l(e)] = \frac{1}{2} \frac{\partial}{\partial n_a} E_0^a[\tau_D] \leq C$$

with C independent of a. Consequently, by (7.5),

$$E^a[\tau(t)] = E^a\left[\int_0^t Q^{\xi_s}[l(e)]ds\right] \le Ct < \infty.$$

The lemma is proved.

It follows from the lemma that τ has a finite, continuous, increasing inverse ϕ with $\phi(0) = 0$. This ϕ will be the boundary local time of the process X.

Now we are ready to define the process X. Let $\omega \in \Omega$ and $t \ge 0$. Define

$$X_t(\omega) = \begin{cases} \xi_0(\omega), & \text{if } t = 0, \\ e_{\phi(t)}(t - \tau(\phi(t-))), & \text{if } \phi(t) \in J(\omega), \\ \xi_{\phi(t)}, & \text{if } \phi(t) \notin J(\omega). \end{cases}$$

We see immediately that the process X has continuous sample paths; $\phi(t)$ increases only when $X_t \in \partial D$ and $X_{\tau(t)} = \xi_t$. We can check directly that the amount of time X spends on the boundary is zero. Indeed, for any $T > 0$,

$$m\{t \in (0, T] : X_t \in M\} = \sum_{s \in J_{\phi(T)-}} l(e_s) + T - \tau(\phi(T)-) = T$$

by the definition of $\tau(t)$.

Let us now define the filtration with respect to which the constructed process X is adapted. Recall that $\{\mathcal{G}_t, t \ge 0\}$ is the filtration associated with the point process of excursions $e = \{e_u, u \ge 0\}$. If $e \in W$ is an excursion path, let e^u denote the path e stopped at u, namely

$$e^u = \{e(t \wedge u), t \ge 0\}.$$

Now define

(7.8) $$\mathcal{F}_t = \mathcal{G}_{\phi(t)-} \vee \sigma\{e_{\phi(t)}^{t-\tau(\phi(t)-)}\}.$$

Since $\phi(t)$ is a $\mathcal{G}_t$-stopping time, the above definition has a meaning. Clearly, X_t is $\mathcal{F}_t$-adapted. Note that we also have

(7.9) $$\mathcal{G}_{\phi(t)-} \subset \mathcal{F}_t \subset \mathcal{G}_{\phi(t)}.$$

Now we can state and prove our main theorem.

THEOREM 7.3. *The process X defined above is an $\mathcal{F}_t$-adapted RBM on D.*

PROOF. Let $f \in C^2(\overline{D})$. We show that the process

(7.10) $$B(f)_t = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta f(X_u)du - \frac{1}{2}\int_0^t \frac{\partial f}{\partial n}(X_u)\phi(du)$$

is an $\mathcal{F}_t$-local martingale.

By the definition of X_t, we have for $\phi(t) \in J$,

$$f(X_t) = f(e_{\phi(t)}(l)) - [f(e_{\phi(t)}(l)) - f(e_{\phi(t)}(t - \tau(\phi(t)-)))].$$

Let

(7.11) $$f_t(u, e, \omega) = [f(e(l)) - f(e(t - \tau(u-)))] \cdot I_{\{l > t - \tau(u-)\}}.$$

Then we can write

(7.12) $$f(X_t) = f(\xi_{\phi(t)}) - f_t(\phi(t), e_{\phi(t)}, \omega).$$

This obviously also holds for $\phi(t) \notin J$ if we make the convention that $f_t(u, \partial, \omega) = 0$.

Let us rewrite the terms on the right side of (7.12). First of all, since ξ is the process generated by $\frac{1}{2}A$, the process

$$M(f)_t = f(\xi_t) - f(\xi_0) - \frac{1}{2}\int_0^t Af(\xi_u)du$$

is an $\mathcal{H}_t$-martingale. Because the σ-field $\mathcal{G}_t$ is generated by $\mathcal{H}_t$ and other events independent of $\bigvee_{t \geq 0} \mathcal{H}_t$, the process $M(f)$ is also a $\mathcal{G}_t$-martingale. Now $\phi(t)$ is $\mathcal{G}_t$-stopping time for fixed t and $t \mapsto \phi(t)$ is increasing. Therefore we have

$$(7.13) \qquad f(\xi_{\phi(t)}) = f(\xi_0) + M(f)_{\phi(t)} + \frac{1}{2}\int_0^{\phi(t)} Af(\xi_u)du$$

and $t \mapsto M(f)_{\phi(t)}$ is a $\mathcal{G}_{\phi(t)}$-local martingale. To rewrite the last term in (7.13), we show first

$$(7.14) \qquad Af(a) = \frac{\partial f}{\partial n}(a) + Q^a\left[\int_0^{l(e)} \Delta f(e(u))du\right].$$

We have

$$(7.15) \qquad Af(a) = \frac{\partial}{\partial n_a}u_f(a) = \frac{\partial}{\partial n_a}(u_f(a) - f(a)) + \frac{\partial f}{\partial n}(a).$$

In the following E_0 will denote the expectation with respect to the ordinary d-dimensional Brownian motion. We have

$$(7.16) \qquad u_f(x) - f(x) = E_0^x[f(B_{\tau_D}) - f(B_0)] = \frac{1}{2}E_0^x\left[\int_0^{\tau_D} \Delta f(B_u)du\right].$$

On the other hand, we can verify easily that for any $g \in C(\overline{D})$,

$$(7.17) \quad Q^a\left[\int_0^l g(e(u))du\right] = \lim_{\varepsilon \to 0} Q^a\left[\int_\varepsilon^l g(e(u))du; l > \varepsilon\right]$$

$$= \lim_{\varepsilon \to 0}\int_D g(\varepsilon, y; a)E_0^y\left(\int_0^{\tau_D} g(B_u)du\right)$$

$$= \frac{1}{2}\lim_{\varepsilon \to 0}\frac{\partial}{\partial n_a}\int_D p_0(\varepsilon, y, a)E_0^y\left[\int_0^{\tau_D} g(B_u)du\right]du$$

$$= \lim_{\varepsilon \to 0}\frac{1}{2}\frac{\partial}{\partial n_a}E_0^a\left[\int_\varepsilon^{\tau_D} g(B_u)du; \tau_D > \varepsilon\right]$$

$$= \frac{1}{2}\frac{\partial}{\partial n_a}E_0^a\left[\int_0^{\tau_D} g(B_u)du\right].$$

The last step can be justified. Equation (7.14) follows from (7.15) to (7.17).

On the other hand, since the process X spends zero amount of time on the boundary, we have by Proposition 7.1,

$$\int_0^{\tau(\phi(t))} \Delta f(X_u)du = \sum_{u \in J_{\phi(t)}}\int_0^{l(e_u)} \Delta f(e_u(\tau))d\tau$$

$$= \mathcal{G}_{\phi(t)}\text{-local martingale} + \int_0^{\phi(t)} Q^{\xi_u}\left[\int_0^{l(e)} \Delta f(e(\tau))d\tau\right]du.$$

Therefore, by (7.13), (7.14) we get

$$(7.18) \qquad f(\xi_{\phi(t)}) = f(X_0) + \mathcal{G}_{\phi(t)}\text{-martingale} + \int_0^t \frac{\partial f}{\partial n}(X_u)\phi(du)$$

$$+ \frac{1}{2} \int_0^{\tau(\phi(t))} \Delta f(X_u)du.$$

To rewrite the second term on the right side of (7.12), let us set

$$(7.19) \qquad g_t(u,e,\omega) = f_t(u,e,\omega) - \frac{1}{2} \int_{t-\tau(u-)}^{l(e)} \Delta f(e(v))dv \cdot I_{\{l>t-\tau(u-)\}}.$$

We claim that for any bounded function $G(e) \in \mathcal{B}_{s-\tau(u-)}(W^a)$ and any $t \geq s$,

$$(7.20) \qquad \int_{W^a} g_t(u,e,\omega)G(e)Q^a(de) = 0.$$

Indeed, since the measure Q^a has the transition density function $p_0(t,x,y)$, and ω and u are regarded as constant in the integration in (7.20), if we replace $t - \tau(u-)$ by t and $s - \tau(u-)$ by s, the left side of (7.20) becomes

$$Q^a \left[G(e) \cdot E_0^{e(s)} \left(f(B_{\tau_D}) - f(B_{t-s}) - \frac{1}{2} \int_{t-s}^{\tau_D} \Delta f(B_u)du; \tau_D > t - s \right) ; l > s \right] = 0,$$

which indeed holds because $E_0^{e(s)}(\cdots) = 0$.

We have

$$f_t(\phi(t), e_{\phi(t)}, \omega) = \sum_{u \in J_{\phi(t)}} f_t(u, e_u, \omega)$$

$$(7.21) \qquad = \sum_{u \in J_{\phi(t)}} g_t(u, e_u, \omega) + \frac{1}{2} \int_{t-\tau(\phi(t)-)}^{l(e_{\phi(t)})} \Delta f(e_{\phi(t)}(u))du$$

$$= \int_0^{\phi(t)+} \int_W g_t(u, e, \omega)\tilde{n}(de\,du) + \frac{1}{2} \int_t^{\tau(\phi(t))} \Delta f(X_u)du.$$

Note that in the summations, only the last term is not zero. Finally, by (7.10) and (7.12), (7.18) and (7.21), we obtain

$$(7.22) \qquad B(f)_t = \mathcal{G}_{\phi(t)}\text{-local martingale} - \int_0^{\phi(t)+} \int_W g_t(u, e, \omega)\tilde{n}(de\,du).$$

It follows from the above relation and (7.9) that to show $B(f)$ is an $\mathcal{F}_t$-local martingale, it suffices to prove the last term in (7.22) is $\mathcal{F}_t$-local martingale, namely for any bounded measurable $H(\omega) \in \mathcal{F}_s$, and any $t \geq s$, we have
(7.23)

$$E \left[\int_0^{\phi(t)+} \int_W g_t(u, e, \omega)\tilde{n}(de\,du)H(\omega) \right] = E \left[\int_0^{\phi(s)+} \int_W g_s(u, e, \omega)\tilde{n}(de\,du)H(\omega) \right].$$

We will now show in fact that both sides reduce to zero. The idea of the following proof is in [6]. By (7.8) and the usual argument, we may assume that $H(\omega) = K(\omega)L(\omega)$, where $K(\omega) \in \mathcal{G}_{\phi(s)-}$ and $L(\omega) = G(e_{\phi(t)}^{s-\tau(\phi(t)-)})$ for a bounded

measurable function G on W. For $K(\omega)$, there exists a $\mathcal{G}_t$-predictable process K_t such that $K(\omega) = K_{\phi(s)}$. Using (7.20) we have

$$E\left[\int_0^{\phi(t)+}\int_W g_t(u,e,\omega)\tilde{n}(de\,du)H(\omega)\right] = E\left[\int_0^{\phi(s)+}\int_W g_t(u,e,\omega)\tilde{n}(de\,du)H(\omega)\right]$$

$$= E\left[\sum_{u\in J_{\phi(s)}} g_t(u,e_u,\omega)K_u(\omega)G(e_u^{s-\tau(u-)})\right]$$

$$= E\left[\int_0^{\phi(s)} K_u(\omega)du\int_W g_t(u,e,\omega)G(e^{s-\tau(u-)})Q^{\xi_u}(de)\right]$$

$$= 0.$$

The same proof applies to the right side of (7.23). Strictly speaking, we have to replace s and t by $s\wedge\tau(T)$ and $t\wedge\tau(T)$ respectively in the above argument to ensure that the proper integrability conditions are satisfied.

We have proved for any $f\in C^2(\overline{D})$, $B(f)$ defined in (7.14) is a $\mathcal{F}_t$-local martingale. By the martingale characterization, we conclude that X is indeed an RBM. We may also start from the fact that $B(f)$ is an $\mathcal{F}_t$-local martingale and prove that there exists an $\mathcal{F}_t$-Brownian motion B_t so that the Skorohod equation (2.6) holds for the triple (X,B,ϕ), thus proving that X is an RBM.

REMARK. We have only considered RBM starting from a point on the boundary. Using an independent copy of killed Brownian motion on D, we can construct RBM starting from any point in $\overline{D}$. We omit the details.

8. Some asymptotic estimates. In this section we verify the following asymptotic formulas used in the preceding sections:

(8.1) $$Q^a[e:l(e)>t]\sim\sqrt{2/\pi t}.$$

(8.2) $$N(a,b)\sim\frac{\Gamma(d/2)}{2\pi^{d/2}}||a-b||^{-d}.$$

For $d\geq 3$,

(8.3) $$E^{a,b}[l(e)]\sim(1/d)||a-b||^2.$$

Space prohibits us to give complete proofs and we are content with indicating the major steps and leave the interested reader to fill out the details. The appendix in [13] is a good reference. Although it only deals with the RBM transition density, the method and most of the estimates there are valid for the killed Brownian motion transition density.

Let D be a bounded domain in R^d $(d\geq 2)$ with C^3 boundary. The fundamental solution $p_0(t,x,y)$ of the heat equation (2.10) can be constructed by the parametrix method. Let

$$\Gamma(t,x,y) = (2\pi t)^{-d/2}e^{-||x-y||^2/2t}$$

be the transition density of the free Brownian motion in R^d. Let ϕ be a C^3 function on $\overline{D}$ such that $0\leq\phi\leq 1$ and $\phi(x)=0$ if $d(x,\partial D)\geq 2\varepsilon$, and $\phi(x)=1$ if $d(x,\partial D)\leq\varepsilon$, for a small but fixed positive ε. For a point $x\in\overline{D}$ close to the boundary, let x_∂ denote the unique point on ∂D such that $d(x,\partial D)=||x-x_\partial||$.

Let $x^* = 2x_\partial - x$. Thus x^* is a point outside of D symmetric to x with respect to the boundary.

As a first approximation of the solution $p_0(t, x, y)$, we set

$$(8.4) \qquad q(t, x, y) = \Gamma(t, x, y) - \phi(x)\Gamma(t, x^*, y).$$

Obviously the boundary condition is satisfied, namely $q(t, x, y) = 0$ if $x \in \partial D$. The parametrix method is to seek a solution $p_0(t, x, y)$ of the following form:

$$(8.5) \qquad p_0(t, x, y) = q(t, x, y) + \int_0^t ds \int_D q(t - s, x, z) f(s, z, y) m(dz).$$

It is not difficult to write down an integral equation for $f(t, x, y)$:

$$f(t, x, y) = \left[\frac{\Delta x}{2} - \frac{\partial}{\partial t} \right] q(t, x, y) - \int_0^t ds \int_D \left[\frac{\Delta}{2} - \frac{\partial}{\partial t} \right] q(t - s, x, z) f(s, z, y) m(dz).$$

This equation can be solved by the iteration method, and we express $f(t, x, y)$ as an absolutely convergent series

$$(8.6) \qquad f(t, x, y) = \sum_{n=0}^{\infty} f_n(t, x, y),$$

where

$$f_0(t, x, y) = \left[\frac{\Delta_x}{2} - \frac{\partial}{\partial t} \right] q(t, x, y),$$

$$f_n(t, x, y) = \int_0^t ds \int_D f_0(t - s, x, z) f_{n-1}(s, z, y) m(dz).$$

Differentiating (8.5) with respect to x, we have

$$(8.7) \quad 2g(t, y; a) = \frac{\partial}{\partial n_a} p_0(t, a, y)$$

$$= \frac{\partial}{\partial n_a} q(t, a, y) + \int_0^t ds \int_D \frac{\partial}{\partial n_a} q(t - s, a, z) f(s, z, y) m(dz)$$

$$= \frac{\partial}{\partial n_a} q(t, a, y) + I_1(t, a, y).$$

For the last term we have the estimate

$$(8.8) \qquad \int_D |I_1(t, a, y)| m(dy) \leq \text{const.}$$

for a constant independent of t and a. A straightforward calculation shows that

$$(8.9) \qquad \int_D \frac{\partial}{\partial n_a} q(t, a, y) m(dy) = 2 \int_D \frac{\partial}{\partial n_a} \Gamma(t, a, y) m(dy) \sim 2\sqrt{\frac{2}{\pi t}}.$$

It follows from (8.7) to (8.9) that

$$\int_D g(t, y; a) m(dy) \sim \sqrt{\frac{2}{\pi t}}.$$

This proves (8.1).

Differentiating (8.7) with respect to y, we obtain

$$4\theta(t,a,b) = \frac{\partial^2}{\partial n_a \partial n_b} p_0(t,a,b)$$

$$(8.10) \qquad = \frac{\partial^2}{\partial n_a \partial n_b} q(t,a,b) + \frac{\partial}{\partial n_b} \int_0^t ds \int_D \frac{\partial}{\partial n_a} q(t-s,a,z) f(s,z,b) m(dz)$$

$$= \frac{\partial^2}{\partial n_a \partial n_b} q(t,a,b) + I_2(t,a,b).$$

For $I_2(t,a,b)$, the following estimate holds:

$$(8.11) \qquad \int_0^\infty |I_2(t,a,b)| dt = O(||a-b||^{-(d-1)}),$$

$$(8.12) \qquad \int_0^\infty t||I_2(t,a,b)|| dt = O(||a-b||^{-(d-3)}).$$

Again by straightforward computations, we have when a and b are close,

$$(8.13) \qquad \int_0^\infty \frac{\partial^2}{\partial n_a \partial n_b} q(t,a,b) dt \sim \frac{2\Gamma(d/2)}{\pi^{d/2}} ||a-b||^{-d},$$

$$(8.14) \qquad \int_0^\infty t \frac{\partial^2}{\partial n_a \partial n_b} q(t,a,b) dt \sim \frac{2\Gamma(d/2)}{d\pi^{d/2}} ||a-b||^{-(d-2)}.$$

(8.12) and (8.14) hold if $d \geq 3$. It follows from (8.9)–(8.14) that

$$N(a,b) = \int_0^\infty \theta(t,a,b) dt \sim \frac{\Gamma(d/2)}{2\pi^{d/2}} ||a-b||^{-d},$$

and if $d \geq 3$

$$E^{a,b}[l(e)] = \frac{\int_0^\infty t\theta(t,a,b) dt}{\int_0^\infty \theta(t,a,b) dt} \sim \frac{1}{d} ||a-b||^2.$$

(8.2) and (8.3) are proved.

9. Concluding remarks. (a) Sato [**12**] proved that under certain conditions, a time homogeneous Markov process is completely determined by its minimal part and its boundary process provided that the process does not spend time on the boundary. In view of Sato's result, it is quite natural to construct a process from a minimal process and a process on the boundary. As we have mentioned earlier, our method can be applied to general diffusions on a compact Riemannian manifold with boundary. Thus at least in this special case we have succeeded in carrying out such a construction. (Recall that the excursion process $P^{a,b}$ in our case depends only on the minimal process.) In contrast to the method in [**15**], the shape of the domain under discussion does not play any role in our approach. This makes it possible to generalize our construction to processes of nondiffusion type with zero sojourn time on the boundary. In the general case, however, the resolvent theory should take the place of the martingale theory used here. This topic will be dealt with in a future publication.

(b) In our construction, we have started with a minimal process X^0 (killed Brownian motion) and a process ξ on the boundary (the boundary process) which are known to be the corresponding parts of a process X. Thus the data we started with is consistent at the beginning. In the case of diffusions, consistency simply

means that the Lévy measure of the boundary process must be related to the minimal process in a certain way. Let us be more precise. Assume L is a smooth elliptic differential operator on D which generates a minimal diffusion X^0 on D. Let $K_L(x, b)$ be the Poisson kernel of L and

$$N_L(a, b) = \frac{1}{2} \frac{\partial}{\partial n_a} K_L(a, b).$$

Now let ξ be a process on the boundary generated by an integrodifferential operator L. Then a necessary and sufficient condition for X^0 and ξ to be the minimal process and boundary process of some diffusion process on $\overline{D}$ is that the Lévy system of ξ have the form $(\mu(a)(\partial/\partial n_a)N_L(a, b), ds)$, where μ is a finite strictly positive function on ∂D. The function μ is immaterial since it can be set to 1 by an equivalent choice of the boundary local time

$$\phi_1(t) = \int_0^t \mu(X_s)\phi_\mu(ds).$$

Now suppose we are given a minimal process X^0 and we want to characterize all possible boundary processes ξ associated with X^0. The answer is that the Lévy kernel, which can be read off from the generator of ξ, must be of the form described above. For example, all oblique RBMs on D will have the same minimal part, and hence their boundary processes have the identical (up to a function μ on the boundary) Lévy system, but the generators of the boundary processes differ one from another by a vector field on ∂D. More precisely, let ν be a smooth vector field on ∂D such that $\nu \cdot n \geq \alpha > 0$, and let A_ν be the infinitesimal generator of the boundary process of the oblique RBM with reflecting direction ν. Then

$$A_\nu(a) = \nu(a) \cdot n(a)A_n + \nu^+(a).$$

Here $\nu^+ = \nu - (\nu \cdot n)n$ is the projection of ν on the tangent plane of the boundary.

(c) The so-called inverse problem is much more difficult to answer. In this problem, we assume that a process on the boundary is given and ask for conditions on minimal processes which can be associated with it. We can ask the following more restricted question: Given a pure jump process ξ on the boundary, is there a minimal process X^0 consistent with ξ so that the resulting diffusion process X is a diffusion with *normal* reflecting boundary condition? If such X^0 exists, is it unique? The problem of uniqueness among diffusions with generators of the form $\nabla \cdot \gamma \nabla$ is equivalent to the physical problem of determining the interior conductivity (γ) of a body by measuring temperature and heat flow at the boundary. Although there are some recent results by analysts towards an affirmative answer to the uniqueness [**14**], the problem as we just stated remains open.

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